RISK ATTRIBUTION[1]

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In this article, we propose a methodology to measure the effective contribution to the total risk and to the tracking error due to asset allocation or selection. We demonstrate that the portfolio historical volatility is explained, first by the assets' volatility and correlation (as for the marginal contribution to risk) and second by the holding's volatility (portfolio's turnover). The results highlight that what matters for effective risk contribution is the time series of contribution to return (holdings at each period times periodic return). Applying these results to effective contribution to tracking error (TE) shows that what matters is the time series of excess return times weight differences. This result is different from marginal contribution to the TE, which depends solely on time series of excess return and not on changes in portfolio's holdings. Our results in risk attribution give an exact decomposition of portfolio's total risk and TE that complement the return attribution analysis. Exact decomposition refers to the fact that the sum of contributions is exactly equal to the portfolio's volatility and TE.

Risk control in asset management, and especially risk measurement, is an increasingly important aspect of performance analysis. Risk control is about monitoring the risk, i.e. the variability (volatility) of the portfolio return, which is beard by the portfolio manager. The topic of this article is to provide an explicit decomposition of portfolio risk that accounts for variable portfolio's weights or active trading. In other words, the focus of attention is the effective contribution to the risk of the portfolio resulting from the investment decisions. As opposed to the marginal contribution to risk, the effective contribution measures the effect of past holdings and trading in an asset, a sector or any other asset class to the total risk of the portfolio. We will analyse the total portfolio risk (standard deviation of the portfolio return) and the tracking error (standard deviation of excess return).

To date, several studies have proposed to explain or decompose the portfolio excess return. Since Fama (1972), authors have split the portfolio performance (return or excess return) into different components, related to the investment process. For example, one of the most used performance attribution model, the Brinson et al. (1986) decomposition, breaks down the portfolio excess return mainly between allocation and selection components. Karnosky and Singer (1994) extended the analysis to a multi-currency portfolio. Others have proposed models dedicated to specific classes of assets. Fong et al. (1983) developed a fixed income attribution model, which split the bond return performance into components specifically based on bond characteristics like the duration, the maturity or the corporate-government spread. More recently, Clarke et al. (2002, 2005) have conducted an analysis allowing portfolio performance to be split between a return-

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forecasting skill (the ability to correctly rank stocks by forecasted return) and an implementation efficiency (the ability to transfer return forecasts into security positions after satisfying constraints). While all these studies help us to understand how the manager has achieved performance, generally they did not tell us which level of risk is beard to realize such excess return. Our risk attribution framework will help to determine the contribution of each carve-out or asset class to the total risk/volatility of the portfolio. These calculations are essential for managers that are responsible for mandates with volatility or tracking error ceilings because they identify the active decisions that did contribute principally to the volatility or tracking error. In addition, in asset management firms, risk attribution is developing fast. According to the PricewaterhouseCoopers (2004) Global Trends in Performance Measurement publication, "v*ery few investment managers have implemented risk attribution in its entirety, although 84% calculate risk measures*".

Major benefits of our risk attribution model are that (1) it takes into account active management (changes in asset positions) and (2) it decomposes total risk and tracking error and attributes it in a way that exactly sums to the retrieve the portfolio's volatility and TE. In addition, our model highlights that when assessing risk attribution, we need to consider holdings volatilities along with asset returns. The consequent result is intuitive: if a manager holds a low volatility asset but trades it actively, the contribution of this asset to the portfolio risk could be large.

Volatility attribution model

To control portfolio's absolute returns, the standard approach is based either on standard deviation over a given period, either on Value at risk calculation. The volatility describes the total risk that the manager has taken and it does not tell us where the numbers come from. Which holdings and trading explain the actual level of volatility? This is the kind of questions that managers or risk officers need to answer. A volatility attribution model aims at identifying the investments decisions that have contributed to the volatility. For example, over the past six months, the attribution model shows that the telecom sector explains 10% of the total volatility of the equity side of the portfolio. It is obvious that the model has to break down the volatility in a way that ensures that the sum of the volatility components is equal to the portfolio volatility.

Before we established the first main result in risk attribution, we will set up our notations and remind the well established results for marginal contribution to risk.

We will note the total risk, $\sigma_P = \sqrt{\frac{1}{T-1}\sum_t^T \left(R_{P,t} - \overline{R}_P\right)^2}$

Where $R_{P,t}$ is the money weighted return of portfolio *P* over a time interval [*t* – 1 , *t*] and $\overline{R}_P$ is the money weighted average return of portfolio *P*. We assume that we can always find a weighting scheme that guarantee that the weighted sum of return is equal to the observed portfolio return, as estimated from start and end value, and contribution and withdrawals.[2]

$$R_{P,t} = \sum_i^I w_{i,t}^P \times R_{i,t}$$

Where $w_{i,t}^P$ is the weighting scheme (based on average invested capital) of asset *i* in portfolio *P* at time *t* – 1.

Since Markowitz, it is well known that the marginal contribution of an asset *i* to the volatility, σ_P, is equal to the correlation of asset i with the portfolio times the asset's volatility. It is also straightforward to show that the weighted sum of marginal contributions is equal to the total portfolio's volatility. Marginal contribution is the derivative of the volatility relative to the holding (see Appendix A for details). Precisely, the marginal contribution $MC_{i,t}$ of asset *i* (or any asset class *i*) to the volatility at time *t* is

$$MC_{i,t}\left(\sigma_{P,t}\right) \equiv \frac{\partial \sigma_{P,t}}{\partial w_{i,t}^P} = \rho\left(R_{i,t}; R_{P,t}\right) \times \sigma_{i,t}$$

Where $\rho\left(R_{i,t}; R_{P,t}\right)$ is the correlation between the returns of *i* and *P* at time *t*.

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[2] This assumption has almost no impact for short period *t* and a small number of transactions. However, if the frequency of transactions is very high and the period t covers several days, this assumption is too restrictive as the total return will be significantly different from the weighted sum of return.

Marginal contribution measures the increase in the actual volatility that is triggered by a 1%-increase in asset *i*. It is also possible to use the marginal contribution to break down the portfolio's volatility:

$$\sum_i^N w_{i,t}^P \times MC_{i,t}(\sigma_{P,t}) = \sigma_P$$

The contribution of asset *i* (or asset class *i*) to the total volatility is equal to the actual holding in *i* (w_i) times its marginal contribution. Using the marginal contribution to decompose the total volatility supposes implicitly that the weights, w_i, have remained constant over the analysis period. This implicit assumption is too restrictive as volatility is often calculated over a one-year period or more. In fact, most portfolios are actively traded and the constant weight hypothesis has to be removed. To allow risk decomposition or attribution with varying weights is the first result that we present in this paper.

The contribution to volatility is obtained by calculating the differential of the volatility function respectively to the historical weights in asset class *i* at each time *t*, w_{it}. As detailed in Appendix B, adapting the formula in order to highlight the important factors in the measure of contribution leads to the first fundamental result:

$$C_{i,T}(\sigma_P) = \frac{\operatorname{cov}(R_P; w_i^P R_i^P)}{\sigma_P} = \rho(R_P; w_i^P R_i^P) \times \sigma(w_i^P R_i^P) \qquad \text{Eq 1}$$

Where $\rho(R_P; w_i^P R_i^P)$ is the correlation between the portfolio return R_P and the contribution of asset *i* to the portfolio return, and $\sigma(w_i^P R_i^P)$ is the volatility of the contribution of asset *i* to the portfolio return. In other words, $\sigma(w_i^P R_i^P)$ is the standard deviation of the time series of contribution to return, $w_{i,t}^P R_{i,t}^P$. Equation 1 shows that risk contribution is a function of return's volatility and trading. In fact, holding an asset with a low volatility could have a large contribution if the manager frequently trades this asset. This result is intuitive; what matters is the contribution to returns (i.e holding times return) and not the single returns. In the case of a portfolio that is periodically rebalanced to keep the weights constant, equation 1 simplifies to the marginal contribution times the weight.

$$C_{i,T}(\sigma_P) = \rho(R_P; w_i^P R_i^P) \times \sigma(w_i^P R_i^P) = w_i^P \times \rho(R_P; R_i^P) \times \sigma(R_i^P)$$

In Appendix B we demonstrate that the sum of contribution given by equation 1 is exactly equal to the portfolio volatility:

$$\begin{aligned}\sigma_P &= \sum_i^N C_{i,T}(\sigma_P) \\ &= \sum_i^N \rho(R_P; w_i^P R_i^P) \times \sigma(w_i^P R_i^P)\end{aligned} \qquad \text{Eq 2}$$

Equation 1 and 2 are fundamental for risk and performance analysts as the sum of contribution is exactly equal to the total portfolio volatility (which results from asset's volatility but also trading). From equation 1, it becomes possible to identify which strategy or which holding has contributed to the actual volatility level. Equation 2 allows explaining the total volatility from the different portfolio's components. Combining our additive model for volatility contribution to a return attribution model would lead to accurate performance attribution, i.e. the comparison of investments in a 2-dimension risk-return space. Additionally, the attribution model helps to monitor and analyse the spread between the risk budgeting and the realised level of risk. For example, a manager that has an annual budget of risk equals to a maximum volatility of 15% could invest in a portfolio that reaches a volatility of 17%. Equation 1 identifies which holding explains the excess volatility over the initial budget.

An illustrative example

We suppose a portfolio with an investment style based on value-growth / large-small caps. During an 18-months period, the manager has taken four active allocation decisions (see table 1). The question is to measure the contribution of each asset class to the total volatility.

The first step is to collect and structure all the required data for calculation. In the case of risk attribution, the required data are:

- Asset returns composing the portfolio and the benchmark for a given frequency corresponding to the portfolio's rebalancing scheme.[3]
- Average invested capitals in each asset *i* for each sub-period
- Benchmark weights

Table 1 shows portfolio's holdings at the end of each month. Return in base currency are equal for portfolio and benchmark because we have assumed that investment is active in style allocation and passive in stock picking. The total portfolio return is only a function of holdings in asset class and not of individual stock return.

[Table 1]

	Style Portfolio Weights				Benchmark Weights				Return in Base Currency					
	Large Cap - Growth	*Small Cap - Growth*	*Large Cap - Value*	*Small Cap - Value*	*Large Cap - Growth*	*Small Cap - Growth*	*Large Cap - Value*	*Small Cap - Value*	*Large Cap - Growth*	*Small Cap - Growth*	*Large Cap - Value*	*Small Cap - Value*	*Bench Return*	*Portf Return*
Jan	20%	22%	30%	28%	24%	30%	26%	20%	-3.5%	11.8%	9.7%	3.0%	5.8%	5.6%
Feb	20%	22%	30%	28%	24%	30%	26%	20%	2.5%	6.6%	7.6%	1.4%	4.8%	4.6%
Mar	22%	20%	30%	28%	26%	28%	26%	20%	-0.3%	8.0%	-5.6%	-3.1%	0.1%	-1.0%
Apr	22%	20%	30%	28%	26%	28%	26%	20%	0.7%	-8.2%	3.4%	7.6%	0.3%	1.6%
May	22%	20%	30%	28%	26%	28%	26%	20%	-0.2%	-8.0%	3.6%	-2.3%	-1.8%	-1.2%
Jun	22%	20%	30%	28%	26%	28%	26%	20%	1.9%	-3.7%	6.7%	6.5%	2.5%	3.5%
Jul	28%	18%	30%	24%	32%	26%	26%	16%	-0.9%	-8.6%	-3.0%	-6.0%	-4.3%	-4.2%
Aug	28%	18%	30%	24%	32%	26%	26%	16%	-2.9%	3.5%	-6.2%	-9.8%	-3.2%	-4.4%
Sep	28%	18%	30%	24%	32%	26%	26%	16%	-1.0%	2.4%	9.7%	-4.1%	2.2%	2.1%
Oct	28%	18%	30%	24%	32%	26%	26%	16%	2.1%	3.9%	-4.0%	-0.8%	0.5%	-0.1%
Nov	30%	15%	30%	25%	34%	23%	26%	17%	2.0%	16.5%	-3.4%	2.6%	4.1%	2.7%
Dec	30%	15%	30%	25%	34%	23%	26%	17%	-3.0%	-11.5%	0.2%	2.6%	-3.2%	-1.9%
Jan	30%	15%	30%	25%	34%	23%	26%	17%	-3.1%	0.8%	4.2%	7.0%	1.4%	2.2%
Feb	30%	15%	30%	25%	34%	23%	26%	17%	3.2%	-0.2%	4.0%	1.1%	2.3%	2.4%
Mar	30%	15%	30%	25%	34%	23%	26%	17%	6.2%	-2.8%	4.7%	-8.8%	1.2%	0.6%
Apr	30%	15%	30%	25%	34%	23%	26%	17%	-1.0%	6.6%	0.4%	0.6%	1.4%	1.0%
May	32%	15%	30%	23%	36%	23%	26%	15%	2.9%	-11.0%	-7.9%	-2.8%	-3.9%	-3.7%
Jun	32%	15%	30%	23%	36%	23%	26%	15%	10.3%	9.8%	0.1%	-0.8%	5.9%	4.6%
Jul	32%	15%	30%	23%	36%	23%	26%	15%	7.2%	7.8%	3.6%	-0.6%	5.2%	4.4%

[3] By this way, the return on asset *i* will be the same in the portfolio and in the benchmark.

During the 18-months period presented in Table 1, the benchmark weights have changed 4 times. And since the portfolio is active in style allocation, the portfolio weights have changed in order keep an under-weight in "Large Cap - Growth" and "Small Cap - Growth" of respectively 4 and 8 percents, and an over-weight in "Large Cap - Value" and "Small Cap - Value" of 4 and 8 percents.

From Table 1, we calculate the return contribution of each style, their volatilities and the correlations between the different styles:[4]

[Table 2]

	Large Cap - Growth	*Small Cap - Growth*	*Large Cap - Value*	*Small Cap - Value*	*Bench*	*Portf*
Return Contribution	7.45%[5]	4.65%	8.83%	-1.26%	22.2%	19.7%
Volatility	3.72%	8.18%	5.32%	4.84%	3.26%	3.07%

For the whole period, we see in Table 2 that the portfolio return is equal to 19.7% and is mainly due to the active style allocation in "Large Cap - Growth" and "Large Cap - Value", which contributes together to 83% of the portfolio performance.[6] For each asset class, we have calculated the return contribution and we verify that the sum of contribution is equal to the total Time Weighted Return (TWR) of the portfolio. Now, we use equation 1 to identify the contribution of each asset class to the portfolio risk (volatility) to complete the return analysis. We therefore need to calculate the volatility of the return contribution of each asset class $\sigma_{CR_i^P}$ and the correlation between the portfolio return and the return contribution each asset class $\rho\left(R_P;CR_i^P\right)$. We verify that the sum of risk contribution is equal to the portfolio volatility. Table 3 and Figure 1 present the results.

[Table 3]

	Large Cap - Growth	*Small Cap - Growth*	*Large Cap - Value*	*Small Cap - Value*	*Portf*	*Bench*
Return Contribution – Portfolio –	7.45%	4.65%	8.83%	-1.26%	**19.7%**	**22.2%**
Risk Contribution – Portfolio –	0.424%	0.824%	1.140%	0.687%	**3.07%**[7]	**3.26%**
Volatility of return contribution – Portfolio –	1.13%	1.42%	1.60%	1.24%		

[4] Remember that we assume that assets volatilities and correlations are constant over the whole period analysed.

[5] Return contribution of an asset class i over period T is equal to $CR_{i,T} = \sum_{t=1}^{T}\left[CR_{i,t} \times \prod_{s=t+1}^{T}\left(1+R_s\right)\right]$. The sum of contribution is equal to the total portfolio return.

[6] (7.45%+8.83%)/19.7%

[7] 3.07% is a monthly volatility. It is equivalent to an annual volatility of $10.63\% = 3.07\% \times \sqrt{12}$

Correlation (portfolio ; return contribution)	0.37	0.58	0.71	0.55

From Table 3, we see that "Large Cap - Growth" has the lowest relative risk contribution to the total volatility while the other class which contributes significantly to the portfolio return, the "Large Cap - Value", has the highest relative risk contribution. From Table 2 we see that the portfolio total volatility is 3.07%. We can easily verify that the sum of each contribution explains the total volatility.

[Figure 1]

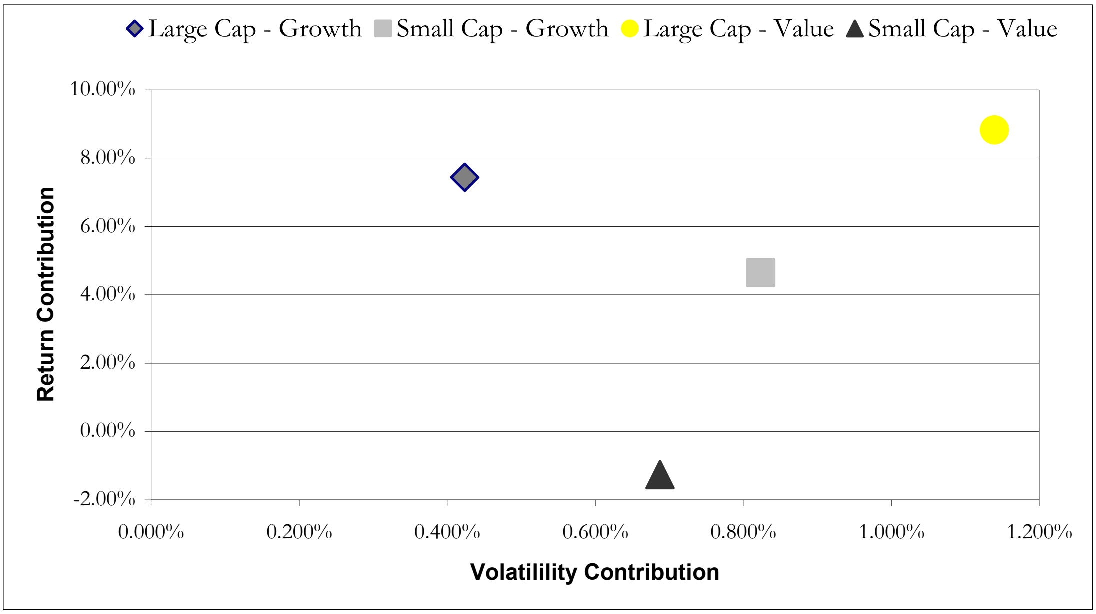

Figure 1 shows the trade-off risk-return for each asset class. The contribution to the volatility on the x-axis takes into account the correlation of an asset class with the portfolio, i.e. its systematic risk and the contribution of trading to the actual level of volatility. Figure 1 highlights which active management decisions add value to the portfolio in the sense that the contribution to the return rewards the additional risk. The analysis shows that asset Large Cap-Value accounts for 45% of the portfolio return (8.83%/19.7%) and for 37.1% of the total portfolio volatility (1.140%/3.07%), while 38% of the total portfolio return comes from Large Cap-Growth with only a risk contribution of 13.8% of the total risk. The active allocation in the class Small Cap-Value adds risk to the portfolio with a negative return. An analysis over a large period of time that confirms consistently these results is helpful to understand where the excess return and risk come from.

Tracking error attribution model

Asset managers are often constraints to keep portfolio's tracking error (TE) below a certain level. In this context, marginal contribution to TE measures the increase (decrease) in the ex-ante TE by over or under-weighting a sector or an asset class. Marginal contributions are generally estimated for sectors or asset classes that reflect the investment process. On an ex-post basis, asset managers have to report on the level of TE and there is a need for a model that attributes or breaks down the realized TE to the different portfolio's components where breakdowns reflects the investment process. That is exactly what we aim to do in this section by using an approach similar to the volatility decomposition. As highlights in the previous section (equation 1), the volatility level depends on the covariance and the trading activity. We will demonstrate that the frequency of trading against the benchmark explains the actual level of TE. The notations that we use for the portfolio TE are:

$$TE_P = \sqrt{\frac{1}{T-1}\sum_t^T \left(R_{P,t} - R_{M,t} - \left(\overline{R_P - R_M}\right)\right)^2}$$

where the indices M stands for the benchmark and P for the portfolio.

As we focus on the tracking error, we assume that the returns are displayed in local currency and that the currency effect are measured and monitored independently[8].

The marginal contribution to TE, also known as the marginal contribution to active risk (Grinhold & Khan 2000) is the first derivative of the TE by the weight difference. The marginal contribution gives the impact of increasing (decreasing) and asset class against the benchmark and is equal to

While marginal contributions are central to the portfolio constructing process, it does not explain how the investment process has produced the realized TE over some historical window. For instance, asset managers that face TE constraints in their investment process are willing to understand the discrepancies between ex-ante risk budgets and realized risk. A model that aims to provides such insights need to take into account the transactions that have occurred over an historical window.

To measure it, we need to calculate the derivative of the tracking error function with respect to the over (under) weights of the portfolio at every time *t*.

According to the detailed analysis in Appendix C, it follows that the **contribution of asset *i* to the tracking error** for a period *T* is equal to:

$$C_{i,T}\left(TE_{P,T}\right) = \rho\left(R_{P,t} - R_{M,t}; w_{i,t}^P R_{i,t}^P - w_{i,t}^M R_{i,t}^M\right) \times \sigma\left(w_{i,t}^P R_{i,t}^P - w_{i,t}^M R_{i,t}^M\right) \qquad \text{Eq 3}$$

Equation 3 highlights that the contribution of a series of over (under) weights in an asset class *i* is equal to the correlation between the portfolio excess return and the asset class *i* excess contribution times the volatility of the excess contribution. This result demonstrates that what matters is the contribution to the return and not the single return. In Appendix 3 we demonstrate that the sum of each contribution to the TE is equal to the portfolio TE, i.e.

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[8] The analysis could be easily extended to address the specific issues of local returns or hedge returns.

$$TE_P = \sum_i^N \rho\left(R_{P,t} - R_{M,t}; w_{i,t}^P R_{i,t}^P - w_{i,t}^M R_{i,t}^M\right) \times \sigma\left(w_{i,t}^P R_{i,t}^P - w_{i,t}^M R_{i,t}^M\right)$$

An illustrative example (follows)

From Table [1], we verify that the portfolio tracking error (TE) is equal to 0.85%. To calculate the contributions of active allocations, according to Equation 3, we need to estimate the volatilities of the excess contribution and the correlation between the excess contribution and the portfolio excess return.

[Table 4]

	Large Cap - Growth	***Small Cap - Growth***	***Large Cap - Value***	***Small Cap - Value***
Excess Return Contribution[9]	-0.91%	-1.92%	1.08%	-0.52%
$\sigma\left(w_{i,t}^P R_{i,t}^P - w_{i,t}^M R_{i,t}^M\right)$	0.15%	0.65%	0.21%	0.39%
$\rho\left(R_{P,t} - R_{M,t}; w_{i,t}^P R_{i,t}^P - w_{i,t}^M R_{i,t}^M\right)$	0.38	0.77	0.36	0.54
$C_{i,T}\left(TE_{P,T}\right)$	0.06%	0.50%	0.08%	0.21%

From the last row of Table [4], we can easily show that the sum of individual risk contribution is equal to the tracking error.

[Figure 2]

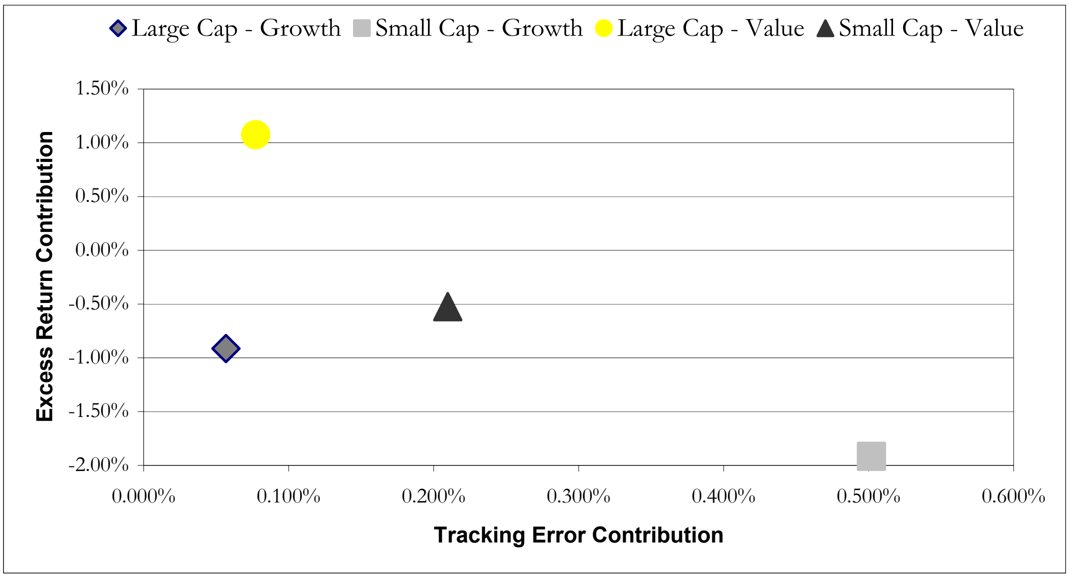

Figure 2 highlights the class Large Cap-Value has the highest contribution to the excess return with a low contribution to the tracking error. The explanation lies in a low correlation with the portfolio excess return and a low volatility of the excess return. We observe that Large Cap-Growth and Small Cap-Growth, while

[9] The contribution to the excess return is given by $ER_{i,T} = \sum_{t=1}^{T}\left[ER_{i,t} \times \prod_{s=t+1}^{T}\left(1 + ER_s\right)\right]$, where ER means the excess return.

contributing positively to the absolute return, increase the tracking error with a negative contribution to the excess return. This simple example shows how an analysis of the contribution to the volatility and to the tracking error leads to a better control of the ex-post risk and return trade-off.

Given these figures, a manager that wants to lower the portfolio TE could change its allocation in order to reduce the under-weighting in “Small Cap - Growth” to 4% (from 8%) and the over-weighting in “Small Cap - Value” to 4% (from 8%). With this new allocation, it is possible to calculate the historical returns of the portfolio. In this case, the portfolio tracking error will be equal to 0.5% instead of 0.85%.[10]

Equation 3 is useful to monitor a portfolio that has TE ceilings. In fact, as the sum of contributions given by equation 3 is equal to the total portfolio TE, it is straightforward to identify which segment, asset or asset class that has contributed to the actual TE level.

Conclusions

In this article, we developed a risk attribution framework to estimate the contribution of an asset class to the total portfolio volatility and tracking error. These results are useful for portfolio performance measurement as the sum of contributions sums exactly to the actual level of volatility or TE. The methodology take into account the trading that has affected the portfolio composition. The first measure gives the risk contribution to the portfolio total volatility (standard deviation of portfolio return). Combine with the contribution to the return, the analysis highlights the trade-off between risk and return that has been achieved and enhance the performance measurement reporting. The second measure identifies the active decisions that did contribute to the actual level of the tracking error. Another major interest of our formalisation is its easy implementation for real active portfolios where rebalancing episodes occur frequently.

[10] Of course, we cannot change the past. But, if we assume that the asset volatilities will remain the same in the near future, our recommendation is defensible.

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Appendix A: Marginal Contribution to the Volatility

We need to calculate the marginal contribution, measured by the derivative of the volatility relative to the holding. Since,

$$\frac{\partial \sigma^2_{P,t}}{\partial w^p_{i,t}} = 2\sigma_{P,t} \frac{\partial \sigma_{P,t}}{\partial w^p_{i,t}}$$

Therefore,

$$\frac{\partial \sigma_{P,t}}{\partial w^P_{i,t}} = \frac{\frac{\partial \sigma^2_{P,t}}{\partial w^P_{i,t}}}{2\sigma_{P,t}} = \frac{\frac{\partial}{\partial w^p_{i,t}}\left(\sum_{i,j}^N w^p_{i,t} \times w^p_{j,t} \times \sigma_{ij}\right)}{2\sigma_{p,t}} = \sum_j^N w^P_{j,t} \times \frac{\sigma_{ij,t}}{\sigma_{P,t}}$$

The **marginal contribution** $C_{i,t}$ of asset i (or any sub-portfolio i) at time t to the volatility can be rewritten as

$$C_{i,t}(\sigma_{P,t}) \equiv \frac{\partial \sigma_{P,t}}{\partial w^P_{i,t}} = \rho(\mathrm{R}_{i,t};\mathrm{R}_{P,t}) \times \sigma_{i,t} \quad \text{Eq A.1}$$

where $\rho(R_{i,t};R_{P,t})$ is the correlation between the returns of i and P at time t.

Appendix B: Contribution to the Volatility

The contribution is linked to the specific holdings in portfolio. For any period t, the relative contribution $C_{i,t}$ of asset i is simply measured by the product of its weight in the portfolio multiplied by its marginal contribution $MC_{i,t}$:

$$C_{i,t}(\sigma_{P,t}) = w_{i,t}^{P} \times MC_{i,t}(\sigma_{P,t})$$

If we need to calculate the marginal contribution of successive holdings over a (recapitulative) period T where changes in weights have occurred, the equation above does not hold. To consider these changes, we need to calculate the differential of the volatility function respectively to the historical weights in asset class i at each time t.

$$MC_{i,T}(\sigma_{P,T}) = d_i\sigma_{P,t} = \frac{\partial\sigma_{P,1}}{\partial w_{i,1}^{P}} dw_{i,1}^{P} + \cdots + \frac{\partial\sigma_{P,t}}{\partial w_{i,t}^{P}} dw_{i,t}^{P} + \cdots + \frac{\partial\sigma_{P,T}}{\partial w_{i,T}^{P}} dw_{i,T}^{P}$$

And given the successive holdings $\left(w_{i,1}^{P}, \cdots, w_{i,T}^{P}\right)$ in asset (sub-portfolio) i, we obtain the total **contribution** over a period T of holding asset (sub-portfolio) i

$$C_{i,T}(\sigma_{P,T}) = \frac{\partial\sigma_{P,1}}{\partial w_{i,1}^{P}} w_{i,1}^{P} + \cdots + \frac{\partial\sigma_{P,t}}{\partial w_{i,t}^{P}} w_{i,t}^{P} + \cdots + \frac{\partial\sigma_{P}}{\partial w_{i,T}^{P}} w_{i,T}^{P} \qquad \text{Eq A.2}$$

To solve this equation and obtain a tractable form for the total relative contribution, we have to determine the derivative of the total variance of the portfolio. Let's first compute for the holding in asset (sub-portfolio) i at period 1

$$\frac{\partial\sigma_{P,1}^{2}}{\partial w_{i,1}^{P}} = \frac{\partial}{\partial w_{i,1}^{P}} \frac{1}{T} \sum_{t}^{T} \left(R_{P,t} - \overline{R_P}\right)^2$$

$$= \frac{1}{T} \sum_{t}^{T} 2\left(R_{P,t} - \overline{R_P}\right) \times \left(\frac{\partial R_{P,t}}{\partial w_{i,1}^{P}} - \frac{\partial \overline{R_P}}{\partial w_{i,1}^{P}}\right)$$

And since

$$\frac{\partial R_{P,t}}{\partial w_{i,1}^{P}} = R_{i,1} \qquad \text{if } t = 1, 0$$

$$\frac{\partial \overline{R_P}}{\partial w_{i,1}^{P}} = \frac{R_{i,1}}{T} \qquad \text{if } t \neq 1$$

which gives

$$\frac{\partial \sigma^2_{P,1}}{\partial w^P_{i,1}} = \frac{2}{T}\left(\left(R_{P,1} - \overline{R_P}\right)\times\left(R_{i,1} - \frac{R_{i,1}}{T}\right) - \sum_{t=2}^{T}\left(R_{P,t} - \overline{R_P}\right)\times\frac{R_{i,1}}{T}\right)$$

$$= \frac{2}{T}\times\left(R_{P,1} - \overline{R_P}\right)\times R_{i,1} - \frac{R_{i,1}}{T}\times\underbrace{\sum_{t=1}^{T}\left(R_{P,t} - \overline{R_P}\right)}_{=0}$$

$$= \frac{2}{T}\times\left(R_{P,1} - \overline{R_P}\right)\times R_{i,1}$$

The derivative of the standard deviation is equal to:

$$\frac{\partial \sigma_{P,1}}{\partial w^P_{i,1}} = \frac{\frac{\partial \sigma^2_{P,1}}{\partial w^P_{i,1}}}{2\sigma_{P,1}} = \frac{1}{T\sigma_{P,1}}\times\left(R_{P,1} - \overline{R_P}\right)\times R_{i,1}$$

The derivatives are calculated for each period t and are introduced in Equation A.2 to give the expression for the **total contribution** of holding an asset i during a period of time T. Since we assume that assets volatilities and correlations are constant over the period analysed, it is equal to:

$$C_{i,T}\left(\sigma_P\right) = \frac{1}{\sigma_p \times T}\sum_t^T\left(R_{p,t} - \overline{R_{p,t}}\right)\times w_{i,t}\times R_{i,t}$$

This formula could be adapted to highlight the important factor in the measure of the relative contribution to the total risk. We remember that $\sum_t^T R_{p,t} = \sum_t^T \overline{R_{p,t}}$. This equality allows us to introduce in the above expression a new term independent of t without changing the result.

Let's define the **contribution** of i to the portfolio return P at time t *:*

$$CR^P_{i,t} = w^P_{i,t}\times R_{i,t}$$

and the average contribution, which is independent of t :

$$\overline{CR^P_i} = \overline{w^P_i \times R_i}$$

We can subtract the average contribution from the contribution at t without changing the result so that

$$C_{i,T}\left(\sigma_P\right) = \frac{1}{\sigma_P \times T}\sum_t^T\left(R_{P,t} - \overline{R_P}\right)\times\left(CR^P_{i,t} - \overline{CR^P_i}\right)$$

This expression shows that the total contribution to the risk depends on the covariance of the portfolio return with the return contribution of asset (sub-portfolio) i. We can rewrite the above equation to emphasize the role of the correlation between the contribution to the return and the portfolio return.

$$C_{i,T}\left(\sigma_P\right) = \frac{\operatorname{cov}\left(R_P; CR_i\right)}{\sigma_P} = \rho\left(R_P; CR^P_i\right)\times\sigma_{CR^P_i} \qquad \text{Eq A.3}$$

Appendix C: Relative Contribution to the tracking error

We need to solve for each elementary period *t* and each asset or sub-portfolio *i*

$$AC_{i,t}\left(TE_{P,t}\right)=\frac{\partial TE_{P,t}}{\partial \Delta w_{i,t}^{P}}$$

where $\Delta w_{i,t}^{P}$ is the difference between the portfolio and the benchmark weights in asset or sub-portfolio *i*. It is often called the active weights:

$$\Delta w_{i,t}^{P}=w_{i,t}^{P}-w_{i,t}^{M}$$

The observed tracking error for a reference period results from several allocation or selection decisions. The marginal contribution of successive holdings in *i* is the differential of the tracking error function:

$$MC_{i,T}\left(TE_{P,T}\right)=d_i TE=\frac{\partial TE}{\partial \Delta w_{i,1}^{p}}d\Delta w_{i,1}^{p}+\cdots+\frac{\partial TE}{\partial \Delta w_{i,t}^{p}}d\Delta w_{i,t}^{p}+\cdots+\frac{\partial TE}{\partial \Delta w_{i,T}^{p}}d\Delta w_{i,T}^{p}$$

Given the sequence of over (under) weights $\left(\Delta w_{i,1}^{p},\cdots,\Delta w_{i,T}^{p}\right)$, the total contribution is

$$C_{i,T}\left(TE_{P,T}\right)=\frac{\partial TE}{\partial \Delta w_{i,1}^{p}}\left(w_{i,1}^{p}-w_{i,1}^{M}\right)+\cdots+\frac{\partial TE}{\partial \Delta w_{i,t}^{p}}\left(w_{i,t}^{p}-w_{i,t}^{M}\right)+\cdots+\frac{\partial TE}{\partial \Delta w_{i,T}^{p}}\left(w_{i,T}^{p}-w_{i,T}^{M}\right) \quad \text{Eq A.4}$$

This equation gives the contribution of a series of active decisions in asset (sub-portfolio) *i*. Each term is the instantaneous increase (decrease) of the tracking error times the magnitude of the over (under) weight decision. To derive a tractable expression, we need to calculate the first derivative of the tracking error for each period *t*. As the tracking error is the volatility of the excess return, we will first calculate the derivative of the variance

$$\frac{\partial TE_{P,t}}{\partial \Delta w_{i,t}}=\frac{\partial TE_{P,t}^{2}}{\partial \Delta w_{i,t}}\Big/2TE_{P,t} \qquad \text{with} \qquad TE_{P,t}^{2}=\frac{1}{T}\sum_{t}^{T}\left(\left(R_{P,t}-R_{M,t}\right)-\left(\overline{R_P-R_M}\right)\right)^{2}$$

To derive an analytical expression of the tracking error contribution, we need to explicit the management process of the portfolio. In fact, we will assume that a manager who follows a benchmark takes active decisions exclusively at level, *i*. All the other management decisions are passive, i.e. replicate the benchmark. As an illustration, we could assume that *i* stand for stocks selection. In this case, the manager over(under) weight stocks without changing its allocation. This assumption implies that the returns of the stocks are the same in the portfolio or the benchmark, i.e. $R_{i,t}^{P}=R_{i,t}^{M}$. Another example would be a manager that over (under) weights a style (large/small caps, value/growth) while selecting the stocks in each style in the same proportion as in the benchmark. In this case, the return of each style will be the same for the portfolio and for the benchmark. Under this assumption, we have

$$
\begin{aligned}
TE_{P,t}^2 &= \frac{1}{T}\sum_t^T\left(\left(\sum_i^N\left(w_{i,t}^P \times R_{i,t}\right)-\left(w_{i,t}^M \times R_{i,t}\right)\right)-\left(\overline{\sum_i^N\left(w_i^P \times R_i\right)-\left(w_i^M \times R_i\right)}\right)\right)^2 \\
&= \frac{1}{T}\sum_t^T\left(\left(\sum_i^N\left(w_{i,t}^P - w_{i,t}^M\right)\times R_{i,t}\right)-\left(\overline{\sum_i^N\left(w_i^P - w_i^M\right)\times R_i}\right)\right)^2
\end{aligned}
$$

The above expression, combined with the same logic as for the contribution to risk, allows us to determine the derivative with respect to asset *i* over period *s* of the tracking error

$$
\begin{aligned}
\frac{\partial TE_{P,s}^2}{\partial \Delta w_{i,s}^P} &= \frac{\partial}{\partial \Delta w_{i,s}^P}\left[\frac{1}{T}\sum_t^T\left(\left(\sum_i^N\left(w_{i,t}^P - w_{i,t}^M\right)\times R_{i,t}\right)-\left(\overline{\sum_i^N\left(w_i^P - w_i^M\right)\times R_i}\right)\right)^2\right] \\
&= \frac{2}{T}\sum_t^T\left(\sum_i^N \Delta w_{i,t}^P \times R_{i,t} - \left(\overline{\sum_i^N \Delta w_i^P \times R_i}\right)\right)\times\left(\frac{\partial\left(\Delta w_{i,t}^P \times R_{i,t}\right)}{\partial \Delta w_{i,s}} - \frac{\partial\left(\overline{\Delta w_i^P \times R_i}\right)}{\partial \Delta w_{i,s}}\right) \\
&= \frac{2}{T}\left(\left(\sum_i^N \Delta w_{i,s}^P \times R_{i,s} - \left(\overline{\sum_i^N \Delta w_i^P \times R_i}\right)\right)\times R_{i,s} - \frac{R_{i,s}}{T}\times\underbrace{\sum_{t=1}^T\left(\sum_i^N \Delta w_{i,t}^P \times R_{i,t} - \left(\overline{\sum_i^N \Delta w_i^P \times R_i}\right)\right)}_{=0}\right) \\
&= \frac{2}{T}\times\left(\sum_i^N \Delta w_{i,s}^P \times R_{i,s} - \left(\overline{\sum_i^N \Delta w_i^P \times R_i}\right)\right)\times R_{i,s} \\
\frac{\partial TE_{P,s}}{\partial \Delta w_{i,s}^P} &= \frac{1}{TE_{P,s}\times T}\times\left(\sum_i^N \Delta w_{i,s}^P \times R_{i,s} - \left(\overline{\sum_i^N \Delta w_i^P \times R_i}\right)\right)\times R_{i,s} \\
&= \frac{1}{TE_{P,s}\times T}\times\left(\left(R_{P,s} - R_{M,s}\right)-\left(\overline{R_P - R_M}\right)\right)\times R_{i,s}
\end{aligned}
$$

Introducing this result into Equation A.4 will give us the **risk contribution to the tracking error** of a series of holdings in *i* over a period *T*

$$
C_{i,T}\left(TE_{P,T}\right) = \frac{1}{TE_{P,T}\times T}\times\sum_t^T\left(\left(\mathrm{R}_{P,t} - \mathrm{R}_{M,t}\right)-\left(\overline{\mathrm{R}_P - \mathrm{R}_M}\right)\right)\times\left(w_{i,t}^P \times \mathrm{R}_{i,t} - w_{i,t}^M \times \mathrm{R}_{i,t}\right)
$$

As before, we verify that $\sum_t^T\left(R_{P,t} - R_{M,t}\right) = \sum_t^T\left(\overline{R_P - R_M}\right)$. The term $w_{i,t}^P \times R_{i,t}$ and $w_{i,t}^M \times \mathrm{R}_{i,t}$ are respectively the contribution of asset *i* at time *t* to the portfolio return *P* $\left(CR_{i,t}^P\right)$ and to the benchmark return *M* $\left(CR_{i,t}^M\right)$. We can now express the contribution to the tracking error as a function of the correlation between the portfolio excess return and the excess contribution of asset *i*

$$C_{i,T}\left(TE_{P,T}\right) = \frac{1}{TE_{P,T} \times T} \times \sum_t^T \left(\left(\mathrm{R}_{P,t} - \mathrm{R}_{M,t} \right) - \left(\overline{\mathrm{R}_P - \mathrm{R}_M} \right) \right) \times \left(\left(CR_{i,t}^P - CR_{i,t}^M \right) - \left(\overline{CR_i^P - CR_i^M} \right) \right)$$

$$= \frac{1}{\sigma\left(\mathrm{R}_{P,t} - \mathrm{R}_{M,t}\right)} \times \frac{1}{T} \times \operatorname{cov}\left(\mathrm{R}_{P,t} - \mathrm{R}_{M,t}; CR_{i,t}^P - CR_{i,t}^M\right)$$

$$C_{i,T}\left(TE_{P,T}\right) = \rho\left(\mathrm{R}_{P,t} - \mathrm{R}_{M,t}; CR_{i,t}^P - CR_{i,t}^M\right) \times \sigma\left(CR_{i,t}^P - CR_{i,t}^M\right) \quad \text{Eq A.5}$$

Equation A.5 is the second main result of this article and it shows that the contribution of a series of over (under) weights in asset *i* is equal to the correlation between the portfolio excess return and asset *i* excess contribution times the volatility of the excess contribution.